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A COMPUTATIONAL PROCEDURE FOR A COVERAGE PROBLEM FOR MULTIPLE SHOTS

by

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August 1968

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ABERDEEN PROVING GROUND, MARYLAND

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A B E R D E E N   P R O V I N G   G R O U N D,   M A R Y L A N D

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ABSTRACT

A frequently occurring problem in weapon systems analysis is the computation of expected fractional damage of an area target engaged by a salvo of area damaging rounds. Of particular interest is the case involving both round to round and occasion to occasion errors. When the number of rounds is large, the available solutions may encounter acute computational difficulty. This report presents a computational procedure, using Jacobi polynomials, which overcomes this difficulty.

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LIST OF SYMBOLS

a,b	Axes of elliptical target in x and y directions
$\bar{f}_N$	Expected fraction of target damaged
N	Number of rounds in salvo
$p_1(K)$	Single shot probability of damaging a target point (x,y) with a round impacting at (u_i,v_i)
$p_2(u_i v_i)$	Distribution of impact points of the i-th round about its aim point
$p_3(x,y)$	Distribution of (x,y)
$p_4(\xi,\eta)$	Distribution of occasion to occasion errors
R_T	Target radius for circular target
(u_i,v_i)	Impact point of the i-th round
$w(z)$	Weighting function in Jacobi polynomials
(x,y)	Target point
(ξ,η)	Common aim point for all weapons
σ	Common value of σ_x and σ_y when $\sigma_x = \sigma_y$
σ_K	Parameter in the damage function, $p_1(K)$
σ_x,σ_y	Standard deviations of the round to round errors
σ_ξ,σ_η	Standard deviation of the occasion to occasion errors
$\Phi_r(z)$	The r-th Jacobi polynomial

I. INTRODUCTION

A frequently occurring problem in weapon systems analysis is the computation of the expected fraction of a target damaged by a salvo of area damaging rounds all aimed at the same aim point. The many variations of this problem are commonly referred to as coverage problems. Of particular interest is the case involving both round to round and occasion to occasion errors. The case involving only round to round errors has been treated by Groves.[1] The more general case involving both types of errors, in the form considered in this report, is contained in an unpublished work by Grubbs.[2] When the number of rounds is large, both Groves' and Grubbs' procedures encounter acute computational difficulty. An analysis of this difficulty and the presentation of an alternate computational procedure employing Jacobi polynomials comprise the substance of this report.

Let $p_1(K) = p_1(K|(u_i,v_i), (x,y))$ be the conditional probability of damaging the target point (x,y) given that the i-th round impacts at point (u_i,v_i) and let $p_2(u_i,v_i)$ be the density function describing the distribution of (u_i,v_i) about the aim point. If all rounds in a volley are identically distributed about the same aim point (ξ,η), then the probability of damage over all impact points and all rounds i, $i=1,2,\ldots,N$, is equal to $1 - (1-z)^N$, where

$$z = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_1(K)\ p_2(u,v)\ du dv. \qquad (1)$$

If one assumes (x,y) is distributed over the target area as $p_3(x,y)$ and the intended aim point (ξ,η) is itself a random variable with density $p_4(\xi,\eta)$, then for N rounds aimed at the common aim point, the expected fraction of the target damaged, $\bar{f}_N$, is given by

* *References are listed on page 23.*

$$\bar{f}_N = \int\int_T \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [1 - (1-z)^N]\ p_3(x,y)\ p_4(\xi,\eta)\ d\xi d\eta dx dy \qquad (2)$$

where T is the target area.

The difficulties arising in the solution of (2) are discussed in Section II. A computational procedure using Jacobi polynomials to overcome these difficulties is developed in Section III. In Section IV an illustrative example is given and numerical results are presented in Section V.

II. NATURE OF THE PROBLEM

The procedure used by Grubbs and Groves to solve (2) consists of first expanding $f_N(z)$ using the binomial expansion

$$f_N(z) = (1-z)^N = \sum_{j=0}^{N} (-1)^j \binom{N}{j} z^j. \qquad (3)$$

Thus (2) can be written as

$$\bar{f}_N = \sum_{j=1}^{N} (-1)^{j+1} \binom{N}{j} G_j, \qquad (4)$$

where

$$G_j = \int\int_T \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} z^j\ p_3(x,y)\ p_4(\xi,\eta)\ d\xi d\eta dx dy. \qquad (5)$$

For some target distributions and for some forms of the distributions $p_3(x,y)$ and $p_4(\xi,\eta)$, G_j can be obtained analytically, see e.g. Reference [3]; however, in most cases approximations are required.

In some problems of interest, $\bar{f}_N$ is desired for large N, e.g., $N \geq 50$. Although $\bar{f}_N$ lies in the interval $0 \leq \bar{f}_N \leq 1$, the individual terms of the series in (4) can become extremely large. As shown in

Section V, Table I., the partial sums of this series may oscillate in sign, initially with increasing magnitude until at some value of the summation index the magnitude begins to decrease. For large N and some J, the quantity $\binom{N}{J} C_J$ may attain an order of magnitude, which exceeds the word length of a digital computer, even when double or triple precision computations are used. When this situation is present, the roundoff error incurred at various stages of the computations exceeds the value of $\overline{f}_N$, hence Equation (4) is of no value in computing the fractional damage, $\overline{f}_N$.

The source of the computational difficulty is the binomial coefficients arising in the expansion of $(1-z)^N$. Accordingly, the approach used in this report is based on finding an alternate expression for $(1-z)^N$. This is done in terms of Jacobi polynomials. In particular, we seek an approximation of the form

$$f_N(z) = (1-z)^N \approx F_M(z) = \sum_{r=0}^{M} a_{rN} \, \Phi_r(z), \; 0 \le z \le 1, \; M \le N \qquad (6)$$

where $\Phi_r(z)$ is the r-th Jacobi polynomial appropriate to the weighting function

$$w(z) = z^{\alpha}(1-z)^{\beta} \;, \; \alpha > -1, \; \beta > -1. \qquad (7)$$

Since $(1-z)^N$ is a polynomial of degree N, Equation (6) is an exact representation when M = N. The computation of $f_N(z)$ using Equation (6) for large M encounters roundoff problems similar to those present when using (3). The Jacobi polynomials, however, provide a weighted least squares approximation and usually lead to high accuracy with only a few terms. This is particulary true for the Chebyshev polynomials, a special case of the Jacobi polynomials. These polynomials have the advantage that each successive approximation is closer to $f_N(z)$, i.e.,

$$|F_{r+1}(z) - f_N(z)| \le |F_r(z) - f_N(z)| \qquad (8)$$

for r=0,1,...,N-1.

A new series for computing $\bar{f}_N$ employing the Jacobi series expansion of $(1-z)^N$ is listed in Section IV. In Section V the behavior of this series is compared to the series derived from the binomial expansion, Equation (4). The objective of the studies performed was to determine whether the new series could be truncated for some M before severe roundoff error arose, and still retain two or three decimal accuracy in $\bar{f}_N$.

III. EXPANSION OF $(1-z)^N$ IN JACOBI POLYNOMIALS

The Jacobi polynomials are orthogonal in the interval $0 \le z \le 1$, the interval of interest for z as defined by (1). The Jacobi polynomial $\Phi_r(z)$ can be generated by the expression

$$\Phi_r(z) = c_r \, z^{-\alpha}(1-z)^{-\beta} \, U_r^r(z), \tag{9}$$

where

$$U_r^j(z) = \frac{d^j}{dz^j}\left[z^{r+\alpha}(1-z)^{r+\beta}\right], \; j=0,1,\ldots,r \tag{10}$$

and c_r is an arbitrary or normalizing constant. See, e.g., Hildebrand.[4] The coefficient a_{rN} in (6) is obtained from the integral

$$a_{rN} = (1/\gamma_r) \int_0^1 w(z) \; (1-z)^N \; \Phi_r(z) \; dz, \tag{11}$$

where

$$\gamma_r = \int_0^1 w(z) \; \Phi_r^2(z) \; dz. \tag{12}$$

The integral in (11) can be integrated by parts. Let $u = (1-z)^N$ and $dv = U_r^r(z)\,dz$. Then $du = -N(1-z)^{N-1}$ and $v = U_r^{r-1}(z)$. Since

$$U_r^j(0) = U_r^j(1) = 0 \quad , \quad j=1,2,\ldots,r-1,$$

$$a_{rN} = (c_r\, N/\gamma_r) \int_0^1 (1-z)^{N-1}\, U_r^{r-1}(z)\, dz. \tag{13}$$

After integrating by parts r times we obtain

$$\begin{aligned} a_{rN} &= (c_r/\gamma_r)\, N(N-1)\ldots(N-r+1) \int_0^1 U_r^{0}(z)\, (1-z)^{N-r}\, dz \\ &= \left[c_r N!/\gamma_r (N-r)!\right] \int_0^1 z^{r+\alpha}(1-z)^{N+\beta}\, dz \\ &= \left[c_r N!/\gamma_r (N-r)!\right] B(r+\alpha+1,\ N+\beta+1), \end{aligned} \tag{14}$$

where $B(x,y)$ is the beta function. As shown by Hildebrand[4], page 271, the quantity γ_r is given by

$$\begin{aligned} \gamma_r &= (-1)^r r!\ A_{rr}\ c_r \int_0^1 z^{r+\alpha}(1-z)^{r+\beta}\, dz \\ &= (-1)^r r!\ A_{rr}\ c_r\ B(r+\alpha+1,\ r+\beta+1), \end{aligned} \tag{15}$$

where A_{rr} is the coefficient of z^r in $\Phi_r(z)$. The coefficients A_{rj} in $\Phi_r(z)$ can be obtained by use of the hypergeometric series. See, e.g., Courant and Hilbert.[5] $\Phi_r(z)$ is given by

$$\Phi_r(z) = \sum_{j=0}^{r} A_{rj}\, z^j, \tag{16}$$

where A_{rj} is obtained by replacing s by $(\alpha+\beta+r+1)$, t by $-r$ and v by $(1+\alpha)$ in the j-th coefficient, H_j, of the hypergeometric series, i.e.,

$$H_j = \prod_{i=1}^{j} (s+i-1)\ (t+i-1)/(v+i-1)i, \quad j=1,2,\ldots,r \tag{17}$$

and $H_0 = 1$. Thus

$$A_{rr} = \frac{(-1)^r (r+\alpha+\beta+1)(r+\alpha+\beta+2)\dots(2r+\alpha+\beta)}{(1+\alpha)(2+\alpha)\dots(r+\alpha)} \tag{18}$$

$$= \frac{(-1)^r \Gamma(2r+\alpha+\beta+1)\Gamma(1+\alpha)}{\Gamma(r+\alpha+\beta+1)} \tag{19}$$

where $\Gamma(x)$ is the Gamma function. Inserting the results of (15) and (19) into (14) and employing the relationship

$$B(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$$

we obtain

$$a_{rN} = \binom{N}{r} \frac{(2r+\alpha+\beta+1)\Gamma(r+\alpha+\beta+1)\Gamma(r+\alpha+1)\Gamma(N+\beta+1)}{\Gamma(1+\alpha)\Gamma(N+r+\alpha+\beta+2)\Gamma(r+\beta+1)} . \tag{20}$$

Equation (20) is made more amenable to computation by noting that

$$a_{oN} = \frac{(\beta+1)(\beta+2)\dots(\beta+N)}{(\alpha+\beta+2)(\alpha+\beta+3)\dots(\alpha+\beta+N+1)} \tag{21}$$

and

$$a_{rN} = a_{r-1,N}\left[\frac{(N-r+1)(r+\alpha+\beta)(r+\alpha)(2r+\alpha+\beta+1)}{r(N+r+\alpha+\beta+1)(r+\beta)(2r+\alpha+\beta-1)}\right]. \tag{22}$$

To avoid an indeterminancy for $r = 1$, $\alpha + \beta = -1$, a_{1N} should be written

$$a_{1N} = a_{oN}\left[\frac{(N-r+1)(r+\alpha)(2r+\alpha+\beta+1)}{r(N+r+\alpha+\beta+1)(r+\beta)}\right].$$

IV. ILLUSTRATIVE EXAMPLE

Let $T = \left\{(x,y) \mid x^2/a^2 + y^2/b^2 \le 1\right\}$,

$$p_1(K|(u_i,v_i), (x,y)) = \exp\left\{(-1/2\sigma_K^2)\left[(x-u_i)^2 + (y-v_i)^2\right]\right\} , \tag{23}$$

$$p_2(u_i,v_i) = (1/2\pi\sigma_x\sigma_y) \exp\left\{(-1/2)\left[(u_i-\xi)^2/\sigma_x^2 + (v_i-\eta)^2/\sigma_y^2\right]\right\} , \tag{24}$$

$$p_3(x,y) = 1/\pi ab \tag{25}$$

and

$$p_4(\xi,\eta) = (1/2\pi\sigma_\xi\sigma_\eta)\exp\left\{(-1/2)\left[(\xi/\sigma_\xi)^2 + (\eta/\sigma_\eta)^2\right]\right\} . \tag{26}$$

For the above example the common aiming point (ξ,η) is distributed about the target center, since the target center is (0,0) and the mean of (ξ,η) is (0,0). By inserting the above representations into Equation (1) it can be shown that z reduces to

$$z = q\exp\left\{(-1/2)\left[(\xi-x)^2/(\sigma_K^2+\sigma_x^2) + (\eta-y)^2/(\sigma_K^2+\sigma_y^2)\right]\right\} , \tag{27}$$

where

$$q = \sigma_K^2/\left[(\sigma_K^2+\sigma_x^2)(\sigma_K^2+\sigma_y^2)\right]^{1/2} . \tag{28}$$

From (2), (5), (6) and (16)

$$\bar{f}_N = 1 - \sum_{r=0}^{N} a_{rN} \sum_{j=0}^{r} A_{rj}\, G_j \tag{29}$$

where G_j is given by Equation (5). By inserting the distributions specified by Equations (23) through (28) into (5), by completing the squares in $z^j p_4(\xi,\eta)$ and integrating over ξ and η, G_j can be shown to reduce to

$$G_j = C(q^j/j)\, I_j; \quad j=1,2,\ldots N, \tag{30}$$

$$G_o = 1,$$

where

$$C = 2\left[(\sigma_K^2+\sigma_x^2)(\sigma_K^2+\sigma_y^2)\right]^{1/2}/ab, \tag{31}$$

$$D_j = \left[(j\sigma_\xi^2+\sigma_K^2+\sigma_x^2)/j\right]^{1/2}, \tag{32}$$

$$E_j = \left[(j\sigma_\eta^2+\sigma_K+\sigma_y)/j\right]^{1/2}, \tag{33}$$

and

$$I_j = (1/2\pi D_j E_j) \int\int_T \exp\left[(-1/2)(x^2/D_j^2 + y^2/E_j^2)\right] dxdy, \tag{34}$$

T being the ellipse $x^2/a^2 + y^2/b^2 = 1$.

Since $A_{rj} = 1$, Equation (29) can be written

$$\bar{f}_N = 1 - \sum_{r=0}^{N} a_{rN} - C \sum_{r=1}^{N} a_{rN} \sum_{j=1}^{r} A_{rj}(q^j/j)\ I_j. \tag{35}$$

The series developed by Grubbs[2], using the binomial expansion, takes the form

$$\bar{f}_N = C \sum_{j=1}^{N} (-1)^{j+1} \binom{N}{j} (q^j/j)\ I_j. \tag{36}$$

When $\sigma_\xi = \sigma_\eta = 0$, i.e., no target location error, this series is identical to that described by Groves[1].

Analytical approximations for I_j are contained in the paper by Grubbs[2]. I_j can also be obtained by numerical methods, e.g., by the method described by Breaux[6] or alternatively as follows: Let

$$\text{erf}\ (t) = 2/\sqrt{\pi} \int_0^t e^{-x^2}\ dx. \tag{37}$$

On many computers this function is as standard as the trigonometric or exponential functions and hence can be used to eliminate one integral in (34). Equation (34) can then be written

$$I_j = \sqrt{2/\pi}/E_j \int_0^b \exp\left[-(1/2)y^2/E_j^2\right] \text{erf}\left[a(1-y^2/b^2)^{1/2}/(\sqrt{2}\ D_j)\right] dy. \tag{38}$$

I_j is now in the form of a single integral and can easily be obtained on a digital computer by use of standard subroutines.

When $\sigma_x = \sigma_y$, $\sigma_\xi = \sigma_\eta = \sigma_{cI}$, and for circular targets, i.e., $a = b = R_T$, we obtain the familiar result

$$I_j = \left[1 - \exp\left(-R_T^2/2\sigma_j^2\right)\right] , \tag{39}$$

where σ_j is the common value of D_j and E_j. When $\sigma_\xi = \sigma_\eta = 0$, I_j takes the form

$$I_j = \left[1 - \exp\left(-\lambda j\right)\right] \tag{40}$$

where
$$\lambda = R_T^2/2(\sigma_K^2+\sigma^2) .$$

To provide an independent method for checking the accuracy and convergence of the series developed in this report, the numerical studies were performed for cases where I_j could be represented by Equation (40). For this case Grubbs[1] series is identical to Groves[1] and can be written

$$\bar{f}_N = C \sum_{j=1}^{N} (-1)^{j+1} \binom{N}{j} (q^j/j)(1-\exp(-\lambda j)). \tag{41}$$

For this case Breaux[7] has found an alternate series

$$\bar{f}_N = C \sum_{j=1}^{N} \left[(1-qe^{-\lambda})^j - (1-q)^j\right]/j. \tag{42}$$

This series is easily summable for all values of the parameters, σ_K, σ, R_T, and N and provides an exact solution for comparison. Note that the above simplifications are made only to provide an exact solution for numerical comparison. The general structure of the computational procedure does not necessitate that I_j be representable by (40) since as pointed out previously the integral form of I_j in Equation (38) can be obtained in fractions of a second by use of standard techniques for numerical quadrature.

V. NUMERICAL RESULTS

The numerical studies were designed to examine the accuracy and rate of convergence of the following series:

$$S_M^{(1)} = C \sum_{j=1}^{M} (-1)^{j+1} \binom{N}{j} (q^j/j)\ I_j \tag{43}$$

$$S_M^{(2)} = 1 - \sum_{r=0}^{M} a_{rN} - C \sum_{r=1}^{M} a_{rN} \sum_{j=1}^{r} A_{rj}(q^j/j)\ I_j \tag{44}$$

$$S_M^{(3)} = - C \sum_{r=1}^{M} a_{rN} \sum_{j=1}^{r} A_{rj}(q^j/j)\ I_j \tag{45}$$

Equation (43) is the series arising from (41), (44) is that arising from (35). By setting z = 0 and M = N in (6) it can be shown that

$$\sum_{r=0}^{N} a_{rN} = 1.$$

Applying this result to Equation (35) leads to the third series $S_M^{(3)}$.

Numerical experiments, see Table I, indicate that when $\bar{f}_N$ is not equal to 0 or 1.0, $S_M^{(2)}$ decreases monotonically and $S_M^{(3)}$ increases monotonically as M increases, both approaching $\bar{f}_N$. A weighted average of the two sequences $S_M^{(2)}$ and $S_M^{(3)}$ would thus seem to offer a more accurate approximation to $\bar{f}_N$. Such a new sequence can be constructed as follows: Let

$$\Delta S_M^{(i)} = \left| S_M^{(i)} - S_{M-1}^{(i)} \right| , \quad i=2,3.$$

Then define

$$S_M^{(4)} = \left[S_M^{(2)}/\Delta S_M^{(2)} + S_M^{(3)}/\Delta S_M^{(3)} \right] \Big/ \left[1/\Delta S_M^{(2)} + 1/\Delta S_M^{(3)} \right]. \tag{46}$$

The weights attached to each sequence are inversely proportional to $\Delta S_M^{(i)}$ since the sequence with smaller $\Delta S_M^{(i)}$ would seemingly be the most accurate. Alternatively, $S_M^{(4)}$ can be viewed as the sequence formed by the intersection of the linear extrapolation of the two sequences $S_M^{(2)}$ and $S_M^{(3)}$.

Results of numerical experiments, not listed here, indicate that best convergence is attained for weight functions having $\alpha = \beta = -1/2$, i.e.,

$$w(z) = z^{-1/2} (1-z)^{-1/2}.$$

For this weight function the Jacobi polynomials reduce to the "shifted" Chebyshev polynomials. These polynomials generally have the best convergence properties, as verified in this case by experiment.

Discussion of Results

The critical parameters effecting the accuracy and convergence of the series solutions are q and N. When $\sigma_\xi = \sigma_\eta = 0$, and $\sigma_x = \sigma_y = \sigma$, Equation (28) can be written

$$\sigma^2 = \sigma_K^2 (1-q)/q. \tag{47}$$

The cases studied were for $\sigma_K = 1.0$, q = .01, .1, .3, .5, .7, .9, N = 50, 100, 150, 200, 300, 400, 500, 1000, and $T_R = 50$ with σ constrained by Equation (47).

A comparison of the convergence of the four series, Equations (43) through (46) is illustrated for a typical case in Table I. The series arising from the binomial expansion is seen to oscillate with extremely large magnitude and does not provide a useful result until most of the 50 terms are added. $S_M^{(2)}$ and $S_M^{(3)}$, on the other hand, approach the true solution monotonically as does $S_M^{(4)}$, the weighted average. It should be noted that the computations were performed on BRLESC (Ballistic Research Laboratories Electronic Scientific Computer) which has a useable word length equivalent of 2^{56} or approximately 10^{17}. The normal word length on most commercial machines is approximately 10^{10}.

For this reason it seems unlikely that $S_M^{(1)}$ could be summed at all; for this case, on a commercial machine, except by use of double precision.

Table II is a listing of the parametric study to compare the accuracy of the four series. For each pair of values (N,q), the five entries in succession are the exact solution, S_M^{o} followed by $S_M^{(i)}$, i=1,2,3,4. The asteriks (*) denote that no useful result could be obtained from $S_M^{(1)}$. The sequences $S_M^{(i)}$, i=1,2,3,4 were terminated either when M = N, or when any intermediate number exceeded 10^{15}. By inspection of Table I it is seen that $S_M^{(4)}$ is the most accurate series with a maximum error of .0002. Note that 31 of the 48 entries could not be computed using the series derived from the binomial expansion. The average computation time for the cases studied was approximately 1.4 seconds per case on BRLESC.

VI. CONCLUSIONS

A computational procedure for determining expected fractional damage for an area target engaged by a salvo of area kill weapons has been presented. The procedure employs Jacobi polynomials and in most cases the successive approximations converge rapidly to the true solution. Two new series solutions have been presented, one increasing monotonically and the other decreasing monotonically with the summation index, both approaching the true solution. A method for averaging the two solutions has also been presented which accelerates the convergence, thereby making the method useful even in extreme cases where numerical difficulties force the termination of the series before convergence of either has been reached.

TABLE I
COMPARISON OF SERIES CONVERGENCE

M	$S_M^{(1)}$	$S_M^{(2)}$	$S_M^{(3)}$	$S_M^{(4)}$
0	*	.92041076 *	*	*
1	.40000000- 1	.76460313	.24969173- 3	.14726572- 2
2	-.40100000	.61806674	.76732188- 3	.29402158- 2
3	.38326000 1	.48542272	.13043077- 2	.32562712- 2
4	-.29745140 2	.37004304	.18414197- 2	.35475147- 2
5	.19267381 3	.27358342	.23304005- 2	.36965226- 2
6	-.10584328 4	.19609150	.27434943- 2	.37690745- 2
7	.50081576 4	.13629484	.30842220- 2	.38387494- 2
8	-.20670582 5	.91970193- 1	.33441129- 2	.38607293- 2
9	.75196713 5	.60424990- 1	.35370562- 2	.38828909- 2
10	-.24317858 6	.38873154- 1	.36729533- 2	.38935206- 2
11	.7040536[illegible] 6	.24742770- 1	.37640344- 2	.38983923- 2
12	-.18557128 7	.15856107- 1	.38231318- 2	.39026238- 2
13	.43318622 7	.10496434- 1	.38593132- 2	.39038179- 2
14	-.32902972 7	.73983605- 2	.38807188- 2	.39048565- 2
15	.18171976 8	.56827172- 2	.38927868- 2	.39052895- 2
16	-.32515228 8	.47729915- 2	.38992717- 2	.39054557- 2
17	.53354859 8	.43114050- 2	.39026279- 2	.39055786- 2
18	-.80459360 8	.40874159- 2	.39042727- 2	.39056078- 2
19	.11169934 9	.39835488- 2	.39050474- 2	.39056285- 2
20	-.14295898 9	.39375548- 2	.39053948- 2	.39056358- 2
21	.16886754 9	.39181221- 2	.39055429- 2	.39056381- 2
22	-.18425666 9	.39102960- 2	.39056036- 2	.39056397- 2
23	.18592284 9			
24	-.17326993 9			
25	.14939647 9			
26	-.11909443 9			
27	.87743003 8			
28	-.59707817 8			
29	.37494008 8			
30	-.21701903 8			
31	.11561148 8			
32	-.56583281 7			
33	.25387117 7			
34	-.10414660 7			
35	.38943608 6			
36	-.13224696 6			
37	.40605800 5			
38	-.11214117 5			
39	.27680621 4			
40	-.60601240 3			
41	.11657346 3			
42	-.19464392 2			
43	.27843075 1			
44	-.32890155			
45	.36381647- 1			
46	.14242527- 2			
47	.40448705- 2			
48	.39005318- 2			
49	.39057258- 2			
50	.39056342- 2			

$\alpha = \beta = -.5$

$N = 50$

$q = .9$

$\sigma_K = 1.0$

$R_T = 50.0$

$\sigma_\xi = \sigma_\eta = 0$

Exact Solution = .39056398-02

*This column of numbers indicates the power of ten by which to multiply the table entry to obtain $S_M^{(1)}$.

TABLE II

PARAMETRIC STUDY OF SERIES ACCURACY

$\alpha = \beta = -1/2, \quad \sigma_\xi = \sigma_\eta = 0, \quad R_T = 50, \quad \sigma_K = 1$

N \ q		.01	.1	.3	.5	.7	.9
50	S^0	.03558	.01758	.00879	.00609	.00473	.00391
	S^1	.03558	.01758	.00879	.00609	.00473	.00391
	S^2	.03558	.01758	.00879	.00609	.00473	.00391
	S^3	.03558	.01758	.00879	.00609	.00473	.00391
	S^4	.03558	.01758	.00879	.00609	.00473	.00391
100	S^0	.06383	.02308	.01062	.00719	.00552	.00452
	S^1	.06383	.02308	.01062	*	*	*
	S^2	.06383	.02308	.01062	.00724	.00602	.00511
	S^3	.06383	.02308	.01062	.00719	.00552	.00451
	S^4	.06383	.02308	.01062	.00719	.00552	.00452
150	S^0	.08673	.02631	.01170	.00784	.00598	.00488
	S^1	.08673	.02631	*	*	*	*
	S^2	.08873	.02631	.01174	.00879	.01053	.01406
	S^3	.08673	.02631	.01170	.00782	.00594	.00480
	S^4	.08673	.02631	.01170	.00784	.00598	.00487
200	S^0	.10566	.02860	.01246	.00829	.00631	.00515
	S^1	.10566	.02860	*	*	*	*
	S^2	.10566	.02860	.01283	.01255	.02035	.02927
	S^3	.10566	.02860	.01246	.00824	.00617	.00494
	S^4	.10566	.02860	.01246	.00829	.00631	.00513
300	S^0	.13522	.03184	.01354	.00894	.00677	.00549
	S^1	.13522	*	*	*	*	*
	S^2	.13522	.03184	.01717	.02855	.05168	.07106
	S^3	.13522	.03184	.01346	.00866	.00631	.00496
	S^4	.13522	.03184	.01354	.00893	.00675	.00547
400	S^0	.15747	.03414	.01431	.00940	.00711	.00575
	S^1	.15747	*	*	*	*	*
	S^2	.15747	.03416	.02605	.05259	.08938	.11640
	S^3	.15747	.03414	.01403	.00878	.00625	.00484
	S^4	.15747	.03414	.01430	.00938	.00706	.00570
500	S^0	.17510	.03592	.01490	.00976	.00736	.00594
	S^1	.17510	*	*	*	*	*
	S^2	.17510	.03604	.03903	.08015	.12731	.15938
	S^3	.17510	.03591	.01433	.00874	.00610	.00468
	S^4	.17510	.03592	.01488	.00971	.00729	.00587
1000	S^0	.23042	.04146	.01675	.01087	.00815	.00656
	S^1	.23042	*	*	*	*	*
	S^2	.23119	.04780	.12643	.21036	.27847	.31823
	S^3	.22958	.04099	.01408	.00791	.00525	.00893
	S^4	.23039	.04145	.01661	.01067	.00793	.00635

REFERENCES

1. A. D. Groves, "A Method for Hand-Computing the Expected Fractional Kill of an Area Target with a Salvo of Area Kill Weapons," Ballistic Research Laboratories Memorandum Report No. 1544, January 1964, AD 438 490.

2. F. E. Grubbs, "Expected Target Damage for a Salvo of Rounds with Non-Circular Normal Delivery Distribution and a Non-Circular Damage Function," an unpublished manuscript.

3. H. K. Weiss, "Methods for Computing the Effectiveness of Area Weapons," BRL Report No. 879, Ballistic Research Laboratories, Aberdeen Proving Ground, Maryland, 1963.

4. F. B. Hildebrand, "Introduction to Numerical Analysis," (McGraw-Hill Book Company, New York, 1956).

5. R. Courant and D. Hilbert, "Methods of Mathematical Physics," (Interscience Publishers, Inc., New York).

6. H. J. Breaux, "Multiple Quadrature Using BRLESC Subroutines," Ballistic Research Laboratories Memorandum Report No. 1877, November 1967.

7. H. J. Breaux, "Summation of a Finite Series Occurring in Kill Probability for a Circular Target Engaged by a Salvo of Area Kill Weapons," Ballistic Research Laboratories Memorandum Report No. 1877, December 1967.